On the general equations of elasticity

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It is known that LAMÉ was the first to transform the equations of elasticity into orthogonal curvilinear coordinates. Such a transformation, put forward by him for the first time in a memorandum published in Vol. 6 of the *Liouville Journal.* (1st Series), was later reproduced in the XVth and XVIth of the *Leçons sur les coordonnées curvilignes.*

The elegant but somewhat prolix calculations of the illustrious French geometer were significantly abbreviated, with a procedure in part different, by C. NEUMANN and by the late and lamented BORCHARDT.

The first of these two authors, in his most interesting Memorandum: *Zur Theorie der Elasticität* (v. 7 of the Berlin Journal, 1859) took up the question again from the beginning, calculating the potential of molecular forces in isotropic bodies, and deducing directly the known equations of the variation of this potential. The simplifications obtained in this work result primarily from certain relations, preliminarily established by the author, among those he calls coefficients of variation of the said potential, before and after the transformation into curvilinear coordinates. (The coefficients are but the expressions by which the variations of the unknown functions are found to be multiplied, in that part of the variation of the integral which is represented by an integral of an equal order of multiplicity).

BORCHARDT also, in the elegant article entitled: *Transformation der Elasticitätsgleichungen in allgemeine orthogonale Coordinaten* (v. 76 of the cited Journal, 1873) reproduced in the *Bulletin des Sciences mathématiques et astronomiques* (v. 8, 1875), founded his deductions upon the variation of the integral which represents the potential of the elastic forces; but the simplification that he attained, derives from both the suppression of certain parts of the integral which are convertible into service integrals and which make no contribution whatsoever to the indefinite equations, and from the direct transformation of the expression which represents the square of the elementary rotation.

Fundamentally, the essential artifice behind the transformation, on the part of all three of the cited authors, consists in the grouping of the three unknown functions and their nine derivatives under four only distinct expressions, which are those representing cubic dilatation and the three components of rotation. In fact LAMÉ begins directly with the Cartesian equations among these four expressions, while NEUMANN and BORCHARDT have pre-ordered the elementary potential in such a form that these expressions alone furnish the terms for the transformed equations.

Now the said artifice, even if it does permit us to arrive more quickly at these equations, as far as the nature of the argument permits, nonetheless leaves obscured a very interesting circumstance which, as it seems, has not yet been noticed by anyone, and which leads to quite unexpected consequences.

To put this point in the greatest clarity, I will begin by establishing *directly* the general equations of elastic equilibrium in orthogonal coordinates of whatever species.

Let q_1, q_2, q_3 be the curvilinear othogonal coordinates of any point whatever in a three dimensional space and let

$$ds^2 = Q_1^2 dq_1^2 + Q_2^2 dq_2^2 + Q_3^2 dq_3^2 \tag{1}$$

be the expression of the square of any linear element whatever in this space.

If you vary the position of each point, you find:

$$\begin{aligned} ds\delta ds = {} & Q_1^2 dq_1 d\delta q_1 + Q_2^2 dq_2 d\delta q_2 + Q_3^2 dq_3 d\delta q_3 \\ & + Q_1 \delta Q_1 dq_1^2 + Q_2 \delta Q_2 dq_2^2 + Q_3 \delta Q_3 dq_3^2. \end{aligned}$$

But we have for $i = 1, 2, 3$

$$d\delta q_i = \frac{\partial \delta q_i}{\partial q_1} dq_1 + \frac{\partial \delta q_i}{\partial q_2} dq_2 + \frac{\partial \delta q_i}{\partial q_3} dq_3;$$

thus, putting

$$
\begin{aligned}
\delta\theta_1 &= \frac{\partial\delta q_1}{\partial q_1} + \frac{\delta Q_1}{Q_1}, \\
\delta\theta_2 &= \frac{\partial\delta q_2}{\partial q_2} + \frac{\delta Q_2}{Q_2}, \\
\delta\theta_3 &= \frac{\partial\delta q_3}{\partial q_3} + \frac{\delta Q_3}{Q_3}, \\
\delta\omega_1 &= \frac{Q_2}{Q_3}\frac{\partial\delta q_2}{\partial q_3} + \frac{Q_3}{Q_2}\frac{\partial\delta q_3}{\partial q_2}, \\
\delta\omega_2 &= \frac{Q_3}{Q_1}\frac{\partial\delta q_3}{\partial q_1} + \frac{Q_1}{Q_3}\frac{\partial\delta q_1}{\partial q_3}, \\
\delta\omega_3 &= \frac{Q_1}{Q_2}\frac{\partial\delta q_1}{\partial q_2} + \frac{Q_2}{Q_1}\frac{\partial\delta q_2}{\partial q_1},
\end{aligned}
\tag{2}
$$

we can write

$$
\frac{\delta ds}{ds} = \lambda_1^2\delta\theta_1 + \lambda_2^2\delta\theta_2 + \lambda_3^2\delta\theta_3 + \lambda_2\lambda_3\delta\omega_1 + \lambda_3\lambda_1\delta\omega_2 + \lambda_1\lambda_2\delta\omega_3, \tag{2$_a$}
$$

where the three quantities λ_1, λ_2, λ_3, defined by

$$
\lambda_i = \frac{Q_i dq_i}{ds},
$$

are the cosines of the angles which the linear element ds makes with the three coordinate lines q_1, q_2, q_3 (thus designating, for purposes of brevity, the lines along which only one of the coordinates varies, q_1 alone, q_2 alone, or q_3).

Now suppose we have a continuous material system, occupying the connected space S, limited by a surface σ, and let this system be in equilibrium under the action: 1st) of externally applied forces to each element of the volume dS and to each element of the surface $d\sigma$; 2nd) of internal forces, in each element dS, developed from the deformation which are determined within the system by external forces. Let that system, *already deformed and in equilibrium*, be one whose points are determined by the coordinates q_1, q_2, q_3.

Let

$$
F_1 dS, \qquad F_2 dS, \qquad F_3 dS
$$

be the components in the q_1, q_2, q_3 directions, of the external force acting on the element of volume dS, and let

$$
\varphi_1 d\sigma, \qquad \varphi_2 d\sigma, \qquad \varphi_3 d\sigma
$$

be the analogous components of the external force aplied to the surface $d\sigma$.

To express the equilibrium conditions of the system, one should imagine that every one of its points $(q_1,\ q_2,\ q_3)$ undergoes a new displacement, by which its coordinates become $q_1 + \delta q_1$, $q_2 + \delta q_2$, $q_3 + \delta q_3$. The work developed by such displacement by the externally acting force on the element of volume dS is

$$(F_1 Q_1 \delta q_1 + F_2 Q_2 \delta q_2 + F_3 Q_3 \delta q_3)\, dS,$$

and that developed by the externally acting force on the surface element $d\sigma$ is

$$(\varphi_1 Q_1 \delta q_1 + \varphi_2 Q_2 \delta q_2 + \varphi_3 Q_3 \delta q_3)\, d\sigma.$$

As for the internal forces, if these do not develop work–unless it be that the imagined translation changes the length of the linear elements–it is manifest that the work developed by these on the element dS cannot have any other form than

$$(\Theta_1 \delta\theta_1 + \Theta_2 \delta\theta_2 + \Theta_3 \delta\theta_3 + \Omega_1 \delta\omega_1 + \Omega_2 \delta\omega_2 + \Omega_3 \delta\omega_3)\, dS$$

since the variation of the linear element depends on $(2)_a$ upon the six quantities $\delta\theta_i$, $\delta\omega_i$ and is cancelled out with these. The six multipliers Θ_i, Ω_i are functions of q_1, q_2, q_3 whose significance we do not need to investigate for the moment. From what we have said to this point, the general equation of equilibrium is the following:

$$\begin{aligned} &\int (F_1 Q_1 \delta q_1 + F_2 Q_2 \delta q_2 + F_3 Q_3 \delta q_3)\, dS \\ &+ \int (\varphi_1 Q_1 \delta q_1 + \varphi_2 Q_2 \delta q_2 + \varphi_3 Q_3 \delta q_3)\, d\sigma \\ &+ \int (\Theta_1 \delta\theta_1 + \Theta_2 \delta\theta_2 + \Theta_3 \delta\theta_3 + \Omega_1 \delta\omega_1 + \Omega_2 \delta\omega_2 + \Omega_3 \delta\omega_3)\, dS = 0. \end{aligned} \tag{3}$$

To extract from this formula the equations of equilibrium, properly speaking, it is necessary to appropriately transform the integrals of the form

$$\int \Theta_i \delta\theta_i dS, \qquad \int \Omega_i \delta\omega_i dS.$$

Beginning with the first, we have (2)

$$\int \Theta_i \delta\theta_i dS = \int \Theta_i \left(\frac{\partial \delta q_i}{\partial q_i} + \frac{\delta Q_i}{Q_i} \right) dS$$

and, for reasons of brevity putting $Q_1\, Q_2\, Q_3 = \nabla$,

$$\int \Theta_i \delta\theta_i dS = \int \nabla\Theta_i \frac{\partial \delta q_i}{\partial q_i} \frac{dS}{\nabla} + \int \frac{\Theta_i \delta Q_i}{Q_i} dS$$

$$= \int \frac{\partial}{\partial q_i} (\nabla\Theta_i \delta q_i) \frac{dS}{\nabla} - \int \left\{ \frac{\partial \nabla\Theta_i}{\partial q_i} \frac{\delta q_i}{\nabla} - \frac{\Theta_i \delta Q_i}{Q_i} \right\} dS.$$

Now from the well-known equation

$$\int \frac{\partial f}{\partial q_i} \frac{dS}{\nabla} = -\int \frac{Q_i f \cos(nq_i)}{\nabla} d\sigma,$$

where "n" is the internal normal to the surface σ, we have

$$\int \frac{\partial}{\partial q_i} \left(\nabla \Theta_i \delta q_i \right) \frac{dS}{\nabla} = -\int Q_i \Theta_i \cos(nq_i) \delta q_i d\sigma,$$

but also

$$\int \Theta_i \delta \theta_i dS = -\int \left\{ \frac{\partial \nabla \Theta_i}{\partial q_i} \frac{\delta q_i}{\nabla} - \frac{\Theta_i \delta Q_i}{Q_i} \right\} dS \\ - \int Q_i \Theta_i \cos(nq_i) \delta q_i d\sigma.$$

Passing to the second integral, we have (2)

$$\int \Omega_1 \delta \omega_1 dS = \int Q_1 \Omega_1 \left(Q_2^2 \frac{\partial \delta q_2}{\partial q_3} + Q_3^2 \frac{\partial \delta q_3}{\partial q_2} \right) \frac{dS}{\nabla} \\ = \int \left\{ \frac{\partial}{\partial q_3} \left(Q_1 Q_2^2 \Omega_1 \delta q_2 \right) + \frac{\partial}{\partial q_2} \left(Q_1 Q_3^2 \Omega_1 \delta q_3 \right) \right\} \frac{dS}{\nabla} \\ - \int \left\{ \frac{\partial \left(Q_1 Q_2^2 \Omega_1 \right)}{\partial q_3} \delta q_2 + \frac{\partial \left(Q_1 Q_3^2 \Omega_1 \right)}{\partial q_2} \delta q_3 \right\} \frac{dS}{\nabla}$$

or, by the theorem we recall,

$$\int \Omega_1 \delta \omega_1 dS = -\int \left\{ \frac{\partial \left(Q_1 Q_2^2 \Omega_1 \right)}{\partial q_3} \delta q_2 + \frac{\partial \left(Q_1 Q_3^2 \Omega_1 \right)}{\partial q_2} \delta q_3 \right\} \frac{dS}{\nabla} \\ - \int \left\{ Q_2 \cos(nq_3) \delta q_2 + Q_3 \cos(nq_2) \delta q_3 \right\} \Omega_1 d\sigma.$$

Analogously are transformed the other two integrals

$$\int \Omega_2 \delta \omega_2 dS, \qquad \int \Omega_3 \delta \omega_3 dS.$$

Substituting in equation (3) the values so transformed of the six integrals

$$\int \Theta_i \delta \theta_i dS, \qquad \int \Omega_i \delta \omega_i dS,$$

we obtain a result of the following form

$$\int \left(S_1 \delta q_1 + S_2 \delta q_2 + S_3 \delta q_3 \right) dS + \int \left(\sigma_1 \delta q_1 + \sigma_2 \delta q_2 + \sigma_3 \delta q_3 \right) d\sigma = 0,$$

the which, because the variation of δq_i is arbitrarily governed, is divided into the three equations

$$S_1 = 0, \qquad S_2 = 0, \qquad S_3 = 0$$

valid at every point of the space S, and into the three equations

$$\sigma_1 = 0, \qquad \sigma_2 = 0, \qquad \sigma_3 = 0$$

valid at every point of the surface σ.

The actual substitutions give the three indefinite equations

$$\begin{aligned}
Q_1 F_1 = \frac{1}{\nabla}\left\{\frac{\partial(\nabla\Theta_1)}{\partial q_1} + \frac{\partial(Q_1^2 Q_3 \Omega_3)}{\partial q_2} + \frac{\partial(Q_1^2 Q_2 \Omega_2)}{\partial q_3}\right\} \\
- \left(\frac{\Theta_1}{Q_1}\frac{\partial Q_1}{\partial q_1} + \frac{\Theta_2}{Q_2}\frac{\partial Q_2}{\partial q_1} + \frac{\Theta_3}{Q_3}\frac{\partial Q_3}{\partial q_1}\right), \\
Q_2 F_2 = \frac{1}{\nabla}\left\{\frac{\partial(Q_2^2 Q_3 \Omega_3)}{\partial q_1} + \frac{\partial(\nabla\Theta_2)}{\partial q_2} + \frac{\partial(Q_2^2 Q_1 \Omega_1)}{\partial q_3}\right\} \\
- \left(\frac{\Theta_1}{Q_1}\frac{\partial Q_1}{\partial q_2} + \frac{\Theta_2}{Q_2}\frac{\partial Q_2}{\partial q_2} + \frac{\Theta_3}{Q_3}\frac{\partial Q_3}{\partial q_2}\right), \\
Q_3 F_3 = \frac{1}{\nabla}\left\{\frac{\partial(Q_3^2 Q_2 \Omega_2)}{\partial q_1} + \frac{\partial(Q_3^2 Q_1 \Omega_1)}{\partial q_2} + \frac{\partial(\nabla\Theta_3)}{\partial q_3}\right\} \\
- \left(\frac{\Theta_1}{Q_1}\frac{\partial Q_1}{\partial q_3} + \frac{\Theta_2}{Q_2}\frac{\partial Q_2}{\partial q_3} + \frac{\Theta_3}{Q_3}\frac{\partial Q_3}{\partial q_3}\right),
\end{aligned} \tag{4}$$

and the three equations at the limits

$$\begin{aligned}
\varphi_1 &= \Theta_1 \cos(nq_1) + \Omega_3 \cos(nq_2) + \Omega_2 \cos(nq_3), \\
\varphi_2 &= \Omega_3 \cos(nq_1) + \Theta_2 \cos(nq_2) + \Omega_1 \cos(nq_3), \\
\varphi_3 &= \Omega_2 \cos(nq_1) + \Omega_1 \cos(nq_2) + \Theta_3 \cos(nq_3).
\end{aligned} \tag{4$_a$}$$

These last equations furnish the definition of the six functions Θ_i, Ω_i. These in fact are applicable to every portion of the system, provided one represent with φ_i the components of the forces which must be applied to the surface of such portion in order to maintain it in equilibrium, when the remaining portion is destroyed. Now for an element $d\sigma_1$ of a surface $q_1 =$ const., we have from $(4)_a$

$$\varphi_1^{(1)} = \Theta_1, \qquad \varphi_2^{(1)} = \Omega_3, \qquad \varphi_3^{(1)} = \Omega_2;$$

for an element $d\sigma_2$ of a surface $q_2 =$ const., we have

$$\varphi_1^{(2)} = \Omega_3, \qquad \varphi_2^{(2)} = \Theta_2, \qquad \varphi_3^{(2)} = \Omega_1;$$

for an element $d\sigma_3$ of a surface $q_3 =$ const., we have

$$\varphi_1^{(3)} = \Omega_2, \qquad \varphi_2^{(3)} = \Omega_1, \qquad \varphi_3^{(3)} = \Theta_3.$$

Thus the quantities Θ_1, Θ_2, Θ_3 represent the unit tensions which are developed *normally* to the surface coordinates $q_1 =$ const., $q_2 =$ const., $q_3 =$ const., and the quantities Ω_1, Ω_2, Ω_3 represent the unit tensions which develop *tangentially* to the said surfaces. The equalities

$$\varphi_2^{(3)} = \varphi_3^{(2)}, \qquad \varphi_3^{(1)} = \varphi_1^{(3)}, \qquad \varphi_1^{(2)} = \varphi_2^{(1)},$$

which result from the preceding values, are those which one ordinarily assumes from the consideration of the elementary tetrahedron.

The equations (4) coincide with those which LAMÉ deduced from the transformation of analogous equations in Cartesian coordinates (*Lecon sur les coordonnées curvilignes*, p. 272). The only difference consists of the fact that LAMÉ introduced into them the derivatives with respect to the arcs instead of (using) Q_1, Q_2, Q_3: but it is very easy to pass from one to the other form by way of formulae which I will indicate below.

But what is more important to note, which is made clearly evident by the process followed here for establishing those equations, is that the space to which they refer is not defined by anything other than the expression (1) of the linear element, without any conditions (being set) for the functions Q_1, Q_2, Q_3. Thus the equations (4), $(4)_a$ possess a much greater generality than their counterparts in Cartesian coordinates, and, in particular, it is useful to note immediately that these *are independent of the postulate of* EUCLID. This fact is intimately connected with what I alluded to at the beginning. But before going further, it is necessary to complete the theory put forward (by us) of the equations of elastic equilibrium.

Let us put

$$\begin{aligned}
\theta_1 &= \frac{\partial x_1}{\partial q_1} + \frac{1}{Q_1}\left(\frac{\partial Q_1}{\partial q_1}x_1 + \frac{\partial Q_1}{\partial q_2}x_2 + \frac{\partial Q_1}{\partial q_3}x_3\right),\\
\theta_2 &= \frac{\partial x_2}{\partial q_2} + \frac{1}{Q_2}\left(\frac{\partial Q_2}{\partial q_1}x_1 + \frac{\partial Q_2}{\partial q_2}x_2 + \frac{\partial Q_2}{\partial q_3}x_3\right),\\
\theta_3 &= \frac{\partial x_3}{\partial q_3} + \frac{1}{Q_3}\left(\frac{\partial Q_3}{\partial q_1}x_1 + \frac{\partial Q_3}{\partial q_2}x_2 + \frac{\partial Q_3}{\partial q_3}x_3\right),\\
\omega_1 &= \frac{Q_2}{Q_3}\frac{\partial x_2}{\partial q_3} + \frac{Q_3}{Q_2}\frac{\partial x_3}{\partial q_2},\\
\omega_2 &= \frac{Q_3}{Q_1}\frac{\partial x_3}{\partial q_1} + \frac{Q_1}{Q_3}\frac{\partial x_1}{\partial q_3},\\
\omega_3 &= \frac{Q_1}{Q_2}\frac{\partial x_1}{\partial q_2} + \frac{Q_2}{Q_1}\frac{\partial x_2}{\partial q_1}.
\end{aligned} \tag{5}$$

Comparing these quantities θ_i, ω_i with the quantities $\delta\theta_i$, $\delta\omega_i$ defined by equations (2), we discern that the second are variations of the first, if we admit that

$$\delta x_i = \delta q_i,$$

and the coordinates q_i to be invariable with respect to δ.

Admitting, as is generally done, that the deformation produced by the external forces be so small that one can treat as differentials the total variations undergone by the coordinates of each point, it is legitimate to understand the initial coordinates to be substituted for the final ones in the functions Q_i, Θ_i, Ω_i, and, considering the quantities x_i as the total increments of the initial coordinates q_i, we can establish the equation

$$\frac{\Delta ds}{ds} = \theta_1\lambda_1^2 + \theta_2\lambda_2^2 + \theta_3\lambda_3^2 + \omega_1\lambda_2\lambda_3 + \omega_2\lambda_3\lambda_1 + \omega_3\lambda_1\lambda_2, \qquad (5)_a$$

analogous to $(2)_a$, to determine the total variation Δds undergone by the element ds during the deformation.

The six quantities θ_i, ω_i (as the preceding $\delta\theta_i$, $\delta\omega_i$) have a most simple geometric significance. In fact, by the effect of the deformation produced by the external forces, the three orthogonal linear elements

$$ds_1 = Q_1 dq_1, \qquad ds_2 = Q_2 dq_2, \qquad ds_3 = Q_3 dq_3$$

of which ds is the resultant, become three linear elements ds_1', ds_2', ds_3' no longer orthogonal but slightly oblique, while ds becomes the resultant ds' of these three new elements. If we then designate by $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ the complements of the plane angles

$$(ds_2',\, ds_3')\,, \qquad (ds_3',\, ds_1')\,, \qquad (ds_1',\, ds_2')\,,$$

we have, from the elementary formula of the resultant,

$$ds'^2 = ds_1'^2 + ds_2'^2 + ds_3'^2 + 2\mathcal{C}_1 ds_2' ds_3' + 2\mathcal{C}_2 ds_3' ds_1' + 2\mathcal{C}_3 ds_1' ds_2'.$$

Putting

$$ds_1' = (1+\alpha_1)\, ds_1, \qquad ds_2' = (1+\alpha_2)\, ds_2, \qquad ds_3' = (1+\alpha_3)\, ds_3,$$
$$ds' = (1+\alpha)\, ds$$

we have from here

$$\alpha = \alpha_1\lambda_1^2 + \alpha_2\lambda_2^2 + \alpha_3\lambda_3^2 + \mathcal{C}_1\lambda_2\lambda_3 + \mathcal{C}_2\lambda_3\lambda_1 + \mathcal{C}_3\lambda_1\lambda_2.$$

But it is evident that we also have

$$\alpha = \frac{ds' - ds}{ds} = \frac{\Delta ds}{ds};$$

such that, comparing the preceding value of α with the formula $(5)_a$, the result is

$$\alpha_i = \theta_i, \qquad \mathcal{C}_i = \omega_i.$$

thus the three quantities θ_i and the three quantities ω_i represent respectively the (relative) lengthening of the sides and the shrinking of the angles of an orthogonal parallelepiped element bounded by the six surface coordinates.

We admit, for well-known reasons, that the virtual work of the internal forces

$$\Theta_1\delta\theta_1 + \Theta_2\delta\theta_2 + \Theta_3\delta\theta_3 + \Omega_1\delta\omega_1 + \Omega_2\delta\omega_2 + \Omega_3\delta\omega_3$$

(with reference to a unit of volume) be an exact variation with respect to the quantities x_i which define the deformation already occurred. The preceding expression, thanks to the substitution of the values of the variations $\delta\theta_i$, $\delta\omega_i$ which are drawn from the formulae (5), becomes:

$$\begin{aligned}
&\sum_{i=1}^{i=3}\left(\frac{\Theta_1}{Q_1}\frac{\partial Q_1}{\partial q_i} + \frac{\Theta_2}{Q_2}\frac{\partial Q_2}{\partial q_i} + \frac{\Theta_3}{Q_3}\frac{\partial Q_3}{\partial q_i}\right)\delta x_i \\
&\quad + \Theta_2\delta\frac{\partial x_1}{\partial q_1} + \frac{Q_1\Omega_3}{Q_2}\delta\frac{\partial x_1}{\partial q_2} + \frac{Q_1\Omega_2}{Q_3}\delta\frac{\partial x_1}{\partial q_3} \\
&\quad + \frac{Q_2\Omega_3}{Q_1}\delta\frac{\partial x_2}{\partial q_1} + \Theta_2\delta\frac{\partial x_2}{\partial q_2} + \frac{Q_2\Omega_1}{Q_3}\delta\frac{\partial x_2}{\partial q_3} \\
&\quad + \frac{Q_3\Omega_2}{Q_1}\delta\frac{\partial x_3}{\partial q_1} + \frac{Q_3\Omega_1}{Q_2}\delta\frac{\partial x_3}{\partial q_2} + \Theta_3\delta\frac{\partial x_3}{\partial q_3}.
\end{aligned}$$

From the form of this expression, it turns out that, if a function Π exists of which this expression be the exact variation, this function cannot but depend upon q_i, upon x_i and upon x_{ij}, which for brevity we can put as

$$x_{ij} = \frac{\partial x_i}{\partial q_j};$$

and properly must be

$$\begin{aligned}
&\frac{\partial \Pi}{\partial x_i} = \sum_{j=1}^{j=3}\frac{\Theta_j}{Q_j}\frac{\partial Q_j}{\partial q_i}, \qquad \frac{\partial \Pi}{\partial x_{ii}} = \Theta_i, \qquad (i = 1, 2, 3) \\
&\frac{\partial \Pi}{\partial x_{12}} = \frac{Q_1\Omega_3}{Q_2}, \qquad \frac{\partial \Pi}{\partial x_{13}} = \frac{Q_1\Omega_2}{Q_3}, \\
&\frac{\partial \Pi}{\partial x_{23}} = \frac{Q_2\Omega_1}{Q_3}, \qquad \frac{\partial \Pi}{\partial x_{21}} = \frac{Q_2\Omega_3}{Q_1}, \\
&\frac{\partial \Pi}{\partial x_{31}} = \frac{Q_3\Omega_2}{Q_1}, \qquad \frac{\partial \Pi}{\partial x_{32}} = \frac{Q_3\Omega_1}{Q_2}.
\end{aligned} \tag{6}$$

Hence the six relations

$$
\begin{aligned}
\frac{Q_3}{Q_2}\frac{\partial\Pi}{\partial x_{23}} &= \frac{Q_2}{Q_3}\frac{\partial\Pi}{\partial x_{32}}(=\Omega_1),\\
\frac{Q_1}{Q_3}\frac{\partial\Pi}{\partial x_{31}} &= \frac{Q_3}{Q_1}\frac{\partial\Pi}{\partial x_{13}}(=\Omega_2),\\
\frac{Q_{23}}{Q_1}\frac{\partial\Pi}{\partial x_{12}} &= \frac{Q_1}{Q_2}\frac{\partial\Pi}{\partial x_{21}}(=\Omega_3),
\end{aligned}
\qquad (6)_a
$$

$$
\frac{\partial\Pi}{\partial x_i} = \sum_{j=1}^{j=3}\frac{1}{Q_j}\frac{\partial Q_j}{\partial q_i}\frac{\partial\Pi}{\partial x_{ii}}, \qquad (i=1,2,3)
$$

the which express that the functions x_1, x_2, x_3 and their first derivatives show up in Π in only six combinations.

$$
\theta_1, \qquad \theta_2, \qquad \theta_3, \qquad \omega_1, \qquad \omega_2, \qquad \omega_3,
$$

however, we also have

$$
\delta\Pi = \frac{\partial\Pi}{\partial\theta_1}\delta\theta_1 + \frac{\partial\Pi}{\partial\theta_2}\delta\theta_2 + \frac{\partial\Pi}{\partial\theta_3}\delta\theta_3 + \frac{\partial\Pi}{\partial\omega_1}\delta\omega_1 + \frac{\partial\Pi}{\partial\omega_2}\delta\omega_2 + \frac{\partial\Pi}{\partial\omega_3}\delta\omega_3
$$

that is to say

$$
\Theta_i = \frac{\partial\Pi}{\partial\theta_i}, \qquad \Omega_i = \frac{\partial\Pi}{\partial\omega_i}, \qquad (i=1,2,3). \qquad (7)
$$

This conclusion could be founded upon the simple observation that the six quantities θ_i, ω_i defined by the equations (5) are not connected to one another by any linear relation independent of x_i, x_{ij}. But the preceding deductions brings out some relations which immediately allow giving the equations (4) and $(4)_a$ a new form. In fact, by virtue of the formulae (6), $(6)_a$, the said equations become

$$
\begin{aligned}
Q_iF_i &= \frac{1}{\nabla}\sum_{j=1}^{j=3}\frac{\partial\left(\nabla\frac{\partial\Pi}{\partial x_{ij}}\right)}{\partial q_j} - \frac{\partial\Pi}{\partial x_i},\\
&\qquad\qquad\qquad\qquad (i=1,2,3)\\
Q_i\varphi_i &= \sum_{j=1}^{j=3}Q_i\frac{\partial\Pi}{\partial x_{ij}}\cos(nq_j),
\end{aligned}
\qquad (8)
$$

and it is precisely in this form that the general equations of elasticity were given by C. NEUMANN, in the cited memorandum. Properly speaking, the functions introduced by NEUMANN (as also by LAMÉ) are not the x_i, but the Q_ix_i, that is they are the components of the displacements: but it is easy to see that if we put

$$
k_i = Q_ix_i
$$

and thence

$$k_{ij} = Q_i x_{ij} + \frac{\partial Q_i}{\partial q_j} x_i,$$

we have, considering as a function of k_i and of k_{ij},

$$\frac{\partial \Pi}{\partial x_i} = \frac{\partial \Pi}{\partial k_i} Q_i + \sum_{j=1}^{j=3} \frac{\partial \Pi}{\partial k_{ij}} \frac{\partial Q_i}{\partial q_j},$$
$$\frac{\partial \Pi}{\partial x_{ij}} = \frac{\partial \Pi}{\partial k_{ij}} Q_i;$$

and by means of these relations, the equations (8) reduce themselves quickly to the following:

$$\left.\begin{aligned} F_i &= \frac{1}{\nabla} \sum_{j=1}^{j=3} \frac{\partial \left(\nabla \frac{\partial \Pi}{\partial k_{ij}} \right)}{\partial q_j} - \frac{\partial \Pi}{\partial k_i}, \\ \varphi_i &= \sum_{j=1}^{j=3} Q_i \frac{\partial \Pi}{\partial k_{ij}} \cos(n q_j) \end{aligned}\right\} \qquad (8)_a$$

which are those of Neumann.

Let us deal now with establishing the equations of elasticity by isotropic means, i.e., by means in which Π has the form

$$\Pi = -\frac{1}{2} \left(A\vartheta^2 + B\varpi \right), \qquad (9)$$

where

$$\vartheta = \theta_1 + \theta_2 + \theta_3,$$
$$\varpi = \omega_1^2 + \omega_2^2 + \omega_3^2 - 4(\theta_2\theta_3 + \theta_3\theta_1 + \theta_1\theta_2).$$

The constants A and B, which depend upon the nature of the medium, are those used by Green (*On the laws of reflexion and refraction of light* etc., 1837). In ordinary theory, the relations between these two constants and the density of the medium represent the squares of the velocities of propagation of the longitudinal and of the transversal waves.

It is worth noting right away that the quantity ϑ, that is the cubic dilatation, has a very simple expression. In fact from the first three equations (5) we easily deduce

$$\vartheta = \frac{1}{\nabla}\left\{\frac{\partial\,(\nabla x_1)}{\partial q_1} + \frac{\partial\,(\nabla x_2)}{\partial q_2} + \frac{\partial\,(\nabla x_3)}{\partial q_3}\right\} \tag{10}$$

From equation (9), by virtue of (7), we deduce:

$$\begin{aligned} \Theta_1 &= -A\vartheta + 2B(\theta_2+\theta_3), \qquad & \Omega_1 &= -B\omega_1, \\ \Theta_2 &= -A\vartheta + 2B(\theta_3+\theta_1), \qquad & \Omega_2 &= -B\omega_2, \\ \Theta_3 &= -A\vartheta + 2B(\theta_1+\theta_2), \qquad & \Omega_3 &= -B\omega_3, \end{aligned} \tag{11}$$

and these values must be substitued in the second members of the equations (4), $(4)_a$.

Such substitution does not offer any difficulty with respect to the equations $(4)_a$.

With respect to the equations (4), it is useful to first of all separate the part multiplied by A from that multiplied by B. As for the first part, we immediately recognice that the second members of the equation (4) reduce themselves to

$$-A\left\{\frac{1}{\nabla}\frac{\partial\,(\nabla\vartheta)}{\partial q_i} + \frac{\vartheta}{\nabla}\frac{\partial\nabla}{\partial q_i}\right\},$$

i.e., to

$$-A\frac{\partial\vartheta}{\partial q_i}, \qquad (i=1,2,3). \tag{α}$$

As for the part which contains the factor B, this part has, in the second member of the first equation (4), the following expression:

$$\begin{aligned} -\frac{B}{\nabla}&\left\{-2\frac{\partial\,[\nabla\,(\theta_2+\theta_3)]}{\partial q_1} + \frac{\partial\,(Q_1^2 Q_2\omega_3)}{\partial q_2} + \frac{\partial\,(Q_1^2 Q_2\omega_2)}{\partial q_3}\right\} \\ &-2B\left\{\frac{\theta_2+\theta_3}{Q_1}\frac{\partial Q_1}{\partial q_1} + \frac{\theta_3+\theta_1}{Q_2}\frac{\partial Q_2}{\partial q_1} + \frac{\theta_1+\theta_2}{Q_3}\frac{\partial Q_3}{\partial q_1}\right\} \end{aligned}$$

or, after some obvious reductions,

$$\begin{aligned} -\frac{2B}{\nabla}&\left\{Q_1\theta_1\frac{\partial\,(Q_2Q_3)}{\partial q_1} - Q_3Q_1\frac{\partial\,(Q_2\theta_2)}{\partial q_1} - Q_1Q_2\frac{\partial\,(Q_3\theta_3)}{\partial q_1}\right. \\ &\left. + \frac{1}{2}\frac{\partial\,(Q_1^2Q_3\omega_3)}{\partial q_2} + \frac{1}{2}\frac{\partial\,(Q_1^2Q_2\omega_2)}{\partial q_3}\right\}. \end{aligned} \tag{$\mathcal{C}$}$$

The direct substitution of the values (5) into this expression $(\mathcal{C})$ would lead to a quite prolix calculation, as Lamé remarks (in the two places cited); precisely to avoid such prolixity, he prefers to begin with the Cartesian equations appropriately prepared to that end. But such a fallback would not be admissible here, after the observations made by us about the greater generality of the equations (4) with respect to the Cartesian. Thus it is necessary to carry out the indicated calculation, the which however, based upon a reasonable induction, can be somewhat abbreviated. Since, in fact, it is known that in ordinary space, the final equations of the isotropy only contain, in the terms multiplied by B, the components of elementary rotation, it is natural to think that these components must also figure in the equations relative to a more general space, since the concept of elementary rotation, according to the definition by W. THOMSON, holds for all space.

In my *Kinematics of fluids* (§ 11) I have already given the general values of the components of rotation in any whatsoever curvilinear coordinates. With orthogonal coordinates, q_1, q_2, q_3 these formulae become

$$\begin{aligned} \vartheta_1 &= \frac{1}{Q_2 Q_3}\left\{\frac{\partial\left(Q_3^2 x_3\right)}{\partial q_2} - \frac{\partial\left(Q_2^2 x_2\right)}{\partial q_3}\right\}, \\ \vartheta_2 &= \frac{1}{Q_3 Q_1}\left\{\frac{\partial\left(Q_1^2 x_1\right)}{\partial q_3} - \frac{\partial\left(Q_3^2 x_3\right)}{\partial q_1}\right\}, \\ \vartheta_3 &= \frac{1}{Q_1 Q_2}\left\{\frac{\partial\left(Q_2^2 x_2\right)}{\partial q_1} - \frac{\partial\left(Q_1^2 x_1\right)}{\partial q_2}\right\}, \end{aligned} \tag{12}$$

where ϑ_1, ϑ_2, ϑ_3 designate *the double components of elementary rotation* which accompanies the deformation of the system or elastic medium. These are also the expression which figure in the equations transformed by LAMÉ, by NEUMANN and by BORCHARDT. The presence, in these formulae, of the products $Q_i^2 x_i$ suggests putting

$$Q_i^2 x_i = K_i$$

and writing the equations (5) in the following form:

$$\begin{aligned} Q_1\theta_1 &= \frac{1}{Q_1}\frac{\partial K_1}{\partial q_1} - \frac{1}{Q_1^2}\frac{\partial Q_1}{\partial q_1}K_1 + \frac{1}{Q_2^2}\frac{\partial Q_1}{\partial q_2}K_2 + \frac{1}{Q_3^2}\frac{\partial Q_1}{\partial q_3}K_3, \\ Q_2\theta_2 &= \frac{1}{Q_2}\frac{\partial K_2}{\partial q_2} + \frac{1}{Q_1^2}\frac{\partial Q_2}{\partial q_1}K_1 - \frac{1}{Q_2^2}\frac{\partial Q_2}{\partial q_2}K_2 + \frac{1}{Q_3^2}\frac{\partial Q_2}{\partial q_3}K_3, \\ Q_3\theta_3 &= \frac{1}{Q_3}\frac{\partial K_3}{\partial q_3} + \frac{1}{Q_1^2}\frac{\partial Q_3}{\partial q_1}K_1 + \frac{1}{Q_2^2}\frac{\partial Q_3}{\partial q_2}K_2 - \frac{1}{Q_3^2}\frac{\partial Q_3}{\partial q_3}K_3, \\ Q_2Q_3\omega_1 &= \frac{\partial K_2}{\partial q_3} + \frac{\partial K_3}{\partial q_2} - 2\left(\frac{1}{Q_2}\frac{\partial Q_2}{\partial q_3}K_2 + \frac{1}{Q_3}\frac{\partial Q_3}{\partial q_2}K_3\right), \\ Q_3Q_1\omega_2 &= \frac{\partial K_3}{\partial q_1} + \frac{\partial K_1}{\partial q_3} - 2\left(\frac{1}{Q_3}\frac{\partial Q_3}{\partial q_1}K_3 + \frac{1}{Q_1}\frac{\partial Q_1}{\partial q_3}K_1\right), \\ Q_1Q_2\omega_3 &= \frac{\partial K_1}{\partial q_2} + \frac{\partial K_2}{\partial q_1} - 2\left(\frac{1}{Q_1}\frac{\partial Q_1}{\partial q_2}K_1 + \frac{1}{Q_2}\frac{\partial Q_2}{\partial q_1}K_2\right). \end{aligned}$$

The substitution of these values in the expression ($\mathcal{C}$) is done quite easily, if we keep the terms which contain the partial derivatives of the first and second orders of the functions of K_i separate from those which contain the functions themselves. The first are grouped together without much difficulty, in the expression

$$-\frac{BQ_1}{Q_2Q_3}\left\{\frac{\partial\left(Q_2\vartheta_2\right)}{\partial q_3}-\frac{\partial\left(Q_3\vartheta_3\right)}{\partial q_2}\right\}. \tag{γ}$$

The second group of terms constitute a homogeneous and linear function of the quantities x_1, x_2, x_3. The coefficients of this function are somewhat complicated; but, with a little attention, these can be easily reduced to a form whose symmetry makes one quickly recognize that the law which governed the composition of all three of the analogous linear functions which go into the equations (4). Putting, that is,

$$\begin{aligned}
H=&\frac{\partial}{\partial q_1}\left\{\frac{1}{Q_1}\frac{\partial\left(Q_2Q_3\right)}{\partial q_1}\right\}+\frac{\partial}{\partial q_2}\left\{\frac{1}{Q_2}\frac{\partial\left(Q_3Q_1\right)}{\partial q_2}\right\}+\frac{\partial}{\partial q_3}\left\{\frac{1}{Q_3}\frac{\partial\left(Q_1Q_2\right)}{\partial q_3}\right\}\\
&-\left\{\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\frac{\partial Q_3}{\partial q_1}+\frac{1}{Q_2}\frac{\partial Q_3}{\partial q_2}\frac{\partial Q_1}{\partial q_2}+\frac{1}{Q_3}\frac{\partial Q_1}{\partial q_3}\frac{\partial Q_2}{\partial q_3}\right\},\\
H_{11}=&\,Q_1\left\{\frac{\partial}{\partial q_2}\left(\frac{1}{Q_2}\frac{\partial Q_3}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{1}{Q_3}\frac{\partial Q_2}{\partial q_3}\right)\right\}+\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\frac{\partial Q_3}{\partial q_1},\\
H_{22}=&\,Q_2\left\{\frac{\partial}{\partial q_3}\left(\frac{1}{Q_3}\frac{\partial Q_1}{\partial q_3}\right)+\frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_3}{\partial q_1}\right)\right\}+\frac{1}{Q_2}\frac{\partial Q_3}{\partial q_2}\frac{\partial Q_1}{\partial q_2},\\
H_{33}=&\,Q_3\left\{\frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{1}{Q_2}\frac{\partial Q_1}{\partial q_2}\right)\right\}+\frac{1}{Q_3}\frac{\partial Q_1}{\partial q_3}\frac{\partial Q_2}{\partial q_3},\\
H_{23}=H_{32}=&\,\frac{1}{Q_2}\frac{\partial Q_1}{\partial q_2}\frac{\partial Q_2}{\partial q_3}+\frac{1}{Q_3}\frac{\partial Q_1}{\partial q_3}\frac{\partial Q_3}{\partial q_2}-\frac{\partial^2 Q_1}{\partial q_2\partial q_3},\\
H_{31}=H_{13}=&\,\frac{1}{Q_3}\frac{\partial Q_2}{\partial q_3}\frac{\partial Q_3}{\partial q_1}+\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\frac{\partial Q_1}{\partial q_3}-\frac{\partial^2 Q_2}{\partial q_3\partial q_1},\\
H_{12}=H_{21}=&\,\frac{1}{Q_1}\frac{\partial Q_3}{\partial q_1}\frac{\partial Q_1}{\partial q_2}+\frac{1}{Q_2}\frac{\partial Q_3}{\partial q_2}\frac{\partial Q_2}{\partial q_1}-\frac{\partial^2 Q_3}{\partial q_1\partial q_2},
\end{aligned} \tag{13}$$

and, taking into account the identiy

$$H_{11}=H_{22}+H_{33}=H, \tag{13_a}$$

we find that the linear function of the x_i relative to the first of the equations (4), can be put under the form

$$-\frac{2B}{Q_2Q_3}\left\{\left(H_{11}-H\right)Q_1x_1+H_{12}Q_2x_2+H_{13}Q_3x_3\right\}$$

or

$$-\frac{B}{Q_2Q_3}\frac{\partial\Phi}{\partial\left(Q_1x_1\right)}. \tag{δ}$$

putting

$$\Phi=\sum_{ij}H_{ij}Q_iQ_jx_ix_j-H\sum_i Q_i^2x_i^2. \tag{14}$$

Gathering up the partial expressions (α), (γ), (δ) and forming the analogous expressions for the second and third of the equations (4), we obtain thus the following indefinite equations of elastic isotropic media:

$$\begin{aligned}
\frac{A}{Q_1}\frac{\partial\vartheta}{\partial q_1}&=\frac{B}{Q_2Q_3}\left\{\frac{\partial\left(Q_2\vartheta_2\right)}{\partial q_3}-\frac{\partial\left(Q_3\vartheta_3\right)}{\partial q_2}\right\}+\frac{B}{Q_1Q_2Q_3}\frac{\partial\Phi}{\partial\left(Q_1x_1\right)}+F_1=0,\\
\frac{A}{Q_2}\frac{\partial\vartheta}{\partial q_2}&=\frac{B}{Q_3Q_1}\left\{\frac{\partial\left(Q_3\vartheta_3\right)}{\partial q_1}-\frac{\partial\left(Q_1\vartheta_1\right)}{\partial q_3}\right\}+\frac{B}{Q_1Q_2Q_3}\frac{\partial\Phi}{\partial\left(Q_2x_2\right)}+F_2=0,\\
\frac{A}{Q_3}\frac{\partial\vartheta}{\partial q_3}&=\frac{B}{Q_1Q_2}\left\{\frac{\partial\left(Q_1\vartheta_1\right)}{\partial q_2}-\frac{\partial\left(Q_2\vartheta_2\right)}{\partial q_1}\right\}+\frac{B}{Q_1Q_2Q_3}\frac{\partial\Phi}{\partial\left(Q_3x_3\right)}+F_3=0.
\end{aligned} \tag{15}$$

As for the equations at the limits $(4)_a$, these do not give rise to any reduction worthy of note, nor do they differ from the ordinary ones, and therefore I do not believe it necessary to transcribe them here at length.

From the form of the equations (15) one deduces that, to form the same equations with the method of the variation of the potential, it is enough to take this potential under the form of

$$-\int\left\{\frac{1}{2}A\vartheta^2+\frac{1}{2}B\left(\vartheta_1^2+\vartheta_2^2+\vartheta_3^2\right)+\frac{B\Phi}{Q_1Q_2Q_3}\right\}dS, \tag{15_a}$$

where we can quickly conclude that the expression

$$\frac{\Phi}{Q_1Q_2Q_3}$$

possesses the same invariant character of the expressions

$$\vartheta \qquad \text{and} \qquad \vartheta_1^2+\vartheta_2^2+\vartheta_3^2$$

Comparing the preceding equations (15) with those given by LAMÉ, and generally admitted, we can see that the first do not agree with the second except when the function Φ be equal to *zero* independently of any hypothesis about the x_i functions, the which, given the identity $(13)_a$, demands that the following hold:

$$H_{11}=0,\qquad H_{22}=0,\qquad H_{33}=0,\qquad H_{23}=0,\qquad H_{31}=0,\qquad H_{12}=0.$$

Now these six equations are precisely those which, in v. 5 of the Journal of Liouville and later in the Vth of the *Lecons sur les coordonnées curvilignes*, LAMÉ himself demonstrated to be necessary because the expression (1) must be a transformation of

$$ds^2 = dx^2 + dy^2 + dz^2$$

or, in other words, because the space in which the elastic means under consideration exists, be Euclidean space. Thus the ordinary equations of isotropy are subordinate to the truth of the postulate of Euclid, while the general equations (4) are, as I have already observed, independent of it.

It is due to this fact, which I alluded to at the beginning of this essay, that the cited authors were forced to adopt various artifices for deducing the equations of isotropy from the general equations, when the form of the linear element, because of the indeterminateness of its coefficients, does not include *a priori* the Euclidean hypothesis. Thus, for example, BORCHARDT profits from the form which takes the integral $(15)_a$, when the coordinates are Cartesian, to directly reduce to (the following form)

$$\frac{1}{2}A\vartheta^2 + \frac{1}{2}B\left(\vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2\right)$$

the quantity under the integral.

If we abandon the Euclidean hypothesis, the equations (15) become the *equations of isotropy in a space of constant curvature.* I say *constant* curvature, because if the curvature of the space were variable, it would not be legitimate to consider *a priori* the coefficients A and B of the expression (9) as a constant quantity. In this regard we can observe that, if the quantity A were variable with q_i, the part corresponding to the term $\frac{1}{2}A\vartheta^2$ of Π in the second members of the equations (15) would be still more simple, that it would be represented, as it is easy to verify, by

$$\frac{1}{Q_i}\frac{\partial\left(A\vartheta\right)}{\partial q_i}, \qquad (i = 1, 2, 3).$$

This is not so for the part relative to the other term $\frac{1}{2}B\varpi$. Now in spaces of constant curvature, the function Φ assumes a most simple form.

In fact the linear element of a space of constant curvature $= \alpha$ can always be put under the form indicated by RIEMANN

$$ds = \frac{\sqrt{dq_1^2 + dq_2^2 + dq_3^2}}{1 + \frac{\alpha}{4}\left(q_1^2 + q_2^2 + q_3^2\right)}$$

which is very useful here because of its symmetry. Putting

$$Q = \frac{1}{1 + \frac{\alpha}{4}\left(q_1^2 + q_2^2 + q_3^2\right)}\left(= Q_1 = Q_2 = Q_3\right),$$

we find (13)

$$H=-3Q^3\alpha,$$

thence

$$H_{11}=H_{22}=H_{33}=-Q^3\alpha,$$

and finally

$$H_{23}=H_{31}=H_{12}=0.$$

As a result of this, when the coordinates q_i are those of RIEMANN, that is those which I called *stereographic* in *Fundamental theory of spaces of constant curvature* (v. 2 of these Annals), we have

$$\frac{\Phi}{Q_1Q_2Q_3}=2\alpha Q^2\left(x_1^2+x_2^2+x_3^2\right)$$

Now the quanatity $Q^2\left(x_1^2+x_2^2+x_3^2\right)$ is the square of the displacement of point $(q_1,\, q_2,\, q_3)$, that is to say, it is that quantity which, with the general orthogonal coordinates referred to in expression (1), is represented by $Q_1^2x_1^2+Q_2^2x_2^2+Q_3^2x_3^2$. Thus in every space which has a constant curvature α relative to orthogonal coordinates, we have

$$\frac{\Phi}{Q_1Q_2Q_3}=2\alpha\left(Q_1^2x_1^2+Q_2^2x_2^2+Q_3^2x_3^2\right) \tag{16}$$

and as a consequence

$$\begin{aligned} H&=-3\alpha Q_1Q_2Q_3,\\ H_{11}=H_{22}=H_{33}&=-\alpha Q_1Q_2Q_3,\\ H_{23}=H_{31}=H_{12}&=0. \end{aligned} \qquad (16)_a$$

These latter six formulae can be transformed, as can the analogous ones of LAMÉ, into just as many geometric relations between the curvatures of the orthogonal surfaces. Denoting, in fact, with $\frac{1}{r_{ij}}$ the geodetic curvature of the line of intersection of the two surfaces $q_i=$ const., $q_j=$ const., when this line is considered as existing on the *first* surface (such that the geodetic curvature of the same line, considered instead as existing on the *second* surface, shall be denoted by $\frac{1}{r_{ji}}$), we have, from well-known formulae, the following relations:

$$\begin{aligned} \frac{\partial Q_1}{\partial q_2}&=\frac{Q_1Q_2}{r_{32}}, & \frac{\partial Q_1}{\partial q_3}&=\frac{Q_1Q_3}{r_{23}},\\ \frac{\partial Q_2}{\partial q_3}&=\frac{Q_2Q_3}{r_{13}}, & \frac{\partial Q_2}{\partial q_1}&=\frac{Q_2Q_1}{r_{31}},\\ \frac{\partial Q_3}{\partial q_1}&=\frac{Q_3Q_1}{r_{21}}, & \frac{\partial Q_3}{\partial q_2}&=\frac{Q_3Q_2}{r_{12}}. \end{aligned}$$

By way of these relations it is possible to eliminate from the last six equations $(16)_a$ all the derivatives of the three functions Q_i, and, if moreover we put

$$Q_i dq_i = ds_i,$$

we can also eliminate from them the functions of Q_i themselves. Operating thus, we find that the three equations

$$H_{11} = H_{22} = H_{33} = -\alpha Q_1 Q_2 Q_3$$

are equivalent to the following

$$\begin{aligned}
\frac{\partial \frac{1}{r_{12}}}{\partial s_2} + \frac{\partial \frac{1}{r_{13}}}{\partial s_3} + \frac{1}{r_{12}^2} + \frac{1}{r_{13}^2} + \frac{1}{r_{21} r_{31}} + \alpha = 0,\\
\frac{\partial \frac{1}{r_{23}}}{\partial s_3} + \frac{\partial \frac{1}{r_{21}}}{\partial s_1} + \frac{1}{r_{23}^2} + \frac{1}{r_{21}^2} + \frac{1}{r_{32} r_{12}} + \alpha = 0,\\
\frac{\partial \frac{1}{r_{31}}}{\partial s_1} + \frac{\partial \frac{1}{r_{32}}}{\partial s_2} + \frac{1}{r_{31}^2} + \frac{1}{r_{32}^2} + \frac{1}{r_{13} r_{23}} + \alpha = 0.
\end{aligned} \tag{16$_b$}$$

As for the other three equations

$$H_{23} = H_{31} = H_{12} = 0,$$

which are identical to three of LAMÉ's, these transform (translate) themselves into the corresponding relations (*Coordonnées curvilignes*, p. 80) among the radii r_{ij}, except that these must naturally be considered as radii of geodetic curvature and not as radii of principal curvature. Moreover, it is to be noted that LAMÉ takes the curvature with the opposite sign.

Designating with α_1, α_2, α_3 the measures of curvature (according to GAUSS) of the three surfaces $q_1 =$ const., $q_2 =$ const., $q_3 =$ const. at the point (q_1, q_2, q_3), and comparing the preceding equations $(16)_b$ with the well-known equations of BONNET, we derive

$$\begin{aligned}
\alpha_1 &= \frac{1}{r_{21} r_{31}} + \alpha,\\
\alpha_2 &= \frac{1}{r_{32} r_{12}} + \alpha,\\
\alpha_3 &= \frac{1}{r_{13} r_{23}} + \alpha.
\end{aligned} \tag{16$_c$}$$

When $\alpha = 0$, that is when the space is Euclidean, the radii of geodetic curvature (r_{21}, r_{31}), (r_{32}, r_{12}), (r_{13}, r_{23}) can be confounded with the radii of principal curvature of the three orthogonal surfaces $q_1 =$ const., $q_2 =$ const., $q_3 =$ const., and the preceding values of α_1, α_2, α_3 coincide with those given by the theorem of GAUSS.

By virtue of the form (16), found for the function Φ, the indefinite equations of the isotropy in a space of constant α can be put definitively in the following form:

$$\begin{aligned}
\frac{A}{Q_1}\frac{\partial\vartheta}{\partial q_1} &= \frac{B}{Q_2Q_3}\left\{\frac{\partial\,(Q_2\vartheta_2)}{\partial q_3} - \frac{\partial\,(Q_3\vartheta_3)}{\partial q_2}\right\} + 4\alpha BQ_1x_1 + F_1 = 0,\\
\frac{A}{Q_2}\frac{\partial\vartheta}{\partial q_2} &= \frac{B}{Q_3Q_1}\left\{\frac{\partial\,(Q_3\vartheta_3)}{\partial q_1} - \frac{\partial\,(Q_1\vartheta_1)}{\partial q_3}\right\} + 4\alpha BQ_2x_2 + F_2 = 0,\\
\frac{A}{Q_3}\frac{\partial\vartheta}{\partial q_3} &= \frac{B}{Q_1Q_2}\left\{\frac{\partial\,(Q_1\vartheta_1)}{\partial q_2} - \frac{\partial\,(Q_2\vartheta_2)}{\partial q_1}\right\} + 4\alpha BQ_3x_3 + F_3 = 0.
\end{aligned} \tag{17}$$

One could foresee *a priori* that the curvature of the space should not be without influence upon the equations of elasticity; but it is without doubt most notable that such influence manifest itself there in such a simple form.

In spite of this simplicity, the theory of elastic media in spaces of constant curvature present most significant differences with respect to the ordinary theory, such as to merit, as it seems to me, a careful study, because of the consequences to which this theory can lead.

I will restrict myself, for the moment, to pointing out summarily some results relative to the case in which the elastic deformation occurs without rotation.

Since in this case the three quantities ϑ_i defined by the equations (12), are zero we can put

$$x_i = \frac{1}{Q_i^2}\frac{\partial U}{\partial q_i} \tag{18}$$

and thence (10)

$$\vartheta = \Delta_2 U \qquad (18)_a$$

where

$$\Delta_2 U = \frac{1}{Q_1Q_2Q_3}\left\{\frac{\partial}{\partial q_1}\left(\frac{Q_2Q_3}{Q_1}\frac{\partial U}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{Q_3Q_1}{Q_2}\frac{\partial U}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{Q_1Q_2}{Q_3}\frac{\partial U}{\partial q_3}\right)\right\}. \qquad (18)_b$$

The equations (17) become in this way

$$\frac{\partial}{\partial q_i}\{A\Delta_2U + 4\alpha BU\} + Q_iF_i = 0, \qquad (i = 1, 2, 3).$$

and show that the forces F must have a potential V, that is, it must be true that

$$F_i = \frac{1}{Q}\frac{\partial V}{\partial q_i}; \qquad (18)_c$$

and with that the said three equations are equivalent to the single

$$A\Delta_2 U + 4\alpha B U + V = 0, \tag{19}$$

in which we must understand as interpenetrated in U, the quantity, independent of q_1, q_2, q_3, which is introduced by the integration.

If we suppose $\vartheta = 0$, that is $\Delta_2 U = 0$, we have from this

$$V = -4\alpha B U, \qquad \Delta_2 V = 0, \tag*{$(19)_a$}$$

and thus we obtain a deformation, without any rotation nor dilatation, in which the force and the displacement have in every point the same (or the opposite) direction and magnitudes constantly proportional. Such a result, which has no counterpart in Euclidean space, presents a singular analogy with certain modern concepts about action of dielectric media (MAXWELL, *Treatise on electricity and magnetism*, v. 1, p. 63). If one admits the equality of direction between the force and the displacement, one must suppose the curvature of the space be negative.

To get a better grasp of the idea, it is worth considering a particular form of the linear element of space with constant curvature α; that is, it is useful to put

$$ds^2 = d\xi^2 + \frac{1}{\alpha}\sin^2\left(\xi\sqrt{\alpha}\right)\left(d\eta^2 + \sin^2\eta d\zeta^2\right), \tag{20}$$

where ξ is the radius vector drawn from a fixed centre to any point whatever of the space, and η, ζ are two angles which determine the direction of this radius, These quantities ξ, η, ζ are the *spherical* coordinates of space with a constant curvature. With such coordinates we have

$$\Delta_2 U = \frac{\alpha}{\sin^2\left(\xi\sqrt{\alpha}\right)}\left\{\frac{1}{\alpha}\frac{\partial}{\partial\xi}\left(\sin^2\left(\xi\sqrt{\alpha}\right)\frac{\partial U}{\partial\xi}\right) + \frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left(\sin\eta\frac{\partial U}{\partial\eta}\right) + \frac{1}{\sin^2\eta}\frac{\partial^2 U}{\partial\zeta^2}\right\} \tag*{$(20)_a$}$$

and we satisfy the equation $\Delta_2 U = 0$ taking

$$U = \mu\cot\left(\xi\sqrt{\alpha}\right), \tag{21}$$

where μ is a constant. This solution corresponds to the ordinary Newtonian elementary potential.

Continuing to designate with x_1, x_2, x_3 the increments of the three variables ξ, η, ζ caused by the elastic deformation, we have in the case of such hypothesis

$$x_1 = \frac{dU}{d\xi} - \frac{\mu\sqrt{\alpha}}{\sin^2\left(\xi\sqrt{\alpha}\right)}, \qquad x_2 = 0, \qquad x_3 = 0,$$

but also (5)

$$\theta_1 = \frac{d^2U}{d\xi^2}, \qquad \theta_2 = \theta_3 = -\frac{1}{2}\frac{d^2U}{d\xi^2},$$
$$\omega_1 = \omega_2 = \omega_3 = 0.$$

The internal tensions of the medium are thus determined (11) by the components

$$\Theta_1 = -2B\frac{d^2U}{d\xi^2}, \qquad \Theta_2 = \Theta_3 = B\frac{d^2U}{d\xi^2},$$
$$\Omega_1 = \Omega_2 = \Omega_3 = 0, \tag{21$_a$}$$

that is to say they are represented by an active force, such as compression or tension, in the direction of the lines of force, and by a force acting in the opposite direction, that is as tension or as compression respectively, in the direction perpendicular to the said lines.

This result also is in harmony with the noted concepts of FARADAY. In actual fact MAXWELL, developing these concepts mathematically (Op. cit., v. 1, p. 128), supposes the compression in the direction of the lines of force and the tension in the normal direction to be equal in absolute value; but recently HELMHOLTZ, in a new dielectric theory (*Monatsberichte* of the Berlin Academy, February, 1881), has already been led, from other considerations, to admit the possibility of a ratio other than unity.

Another simple solution to the equation $\Delta_2 U = 0$, considered under the form of $(20)_a$, is given by

$$U = \mu\zeta, \tag{22}$$

where μ is a constant. This solution corresponds, or rather is identical, to the ordinary electromagnetic potential of a rectilinear current which runs along the polar axis $\eta = 0$. For the calculation of the internal tensions which occur in this case, however, another form of the linear element is more useful, and that is the following:

$$ds^2 = du^2 + \cos^2\left(u\sqrt{\alpha}\right)dz^2 + \frac{1}{\alpha}\sin^2\left(u\sqrt{\alpha}\right)d\zeta^2,$$

where u is the distance of any point whatever of the space from a fixed axis, z is the distance of the foot of this perpendicular from a fixed point of the same axis, ζ is the angle that the plane conducted by the fixed axis and by the arbitrary point (that is, by the "any point whatsoever' chosen) makes with a fixed plane. These quantities u, z, ζ are the *cylindrical* coordinates of space with a constant curvature.

By means of these coordinates we find (supposing that the current runs along the fixed axis $u = 0$)

$$x_1 = 0, \qquad x_2 = 0, \qquad x_3 = \frac{\mu\alpha}{\sin^2\left(u\sqrt{\alpha}\right)},$$

and hence from the equations (5) one draws

$$\theta_1 = \theta_2 = \theta_3 = 0,$$
$$\omega_1 = 0, \qquad \omega_2 = -\frac{2\mu\alpha\cos(u\sqrt{\alpha})}{\sin^2(u\sqrt{\alpha})}, \qquad \omega_3 = 0.$$

The internal tensions of the medium are thus determined (11) by the components

$$\Theta_1 = \Theta_2 = \Theta_3 = 0,$$
$$\Omega_1 = 0, \qquad \Omega_2 = \frac{2B\mu\alpha\cos(u\sqrt{\alpha})}{\sin^2(u\sqrt{\alpha})}, \qquad \Omega_3 = 0, \tag{22$_a$}$$

that is to say are represented uniquely by a force of torsion around the line $u =$ const., $z =$ const., or around the lines which are in one and the same plane with the line along which the current runs, and have their points equidistant from the line. If, maintaining the particular hypothesis (18), we wish to consider the vibratory motion of an elastic medium, in the absence of any accelerating external force, we must admit that the function U depends, besides upon the coordinates q_i, upon the time t, and put

$$F_i = -\rho Q_i \frac{\partial^2 x_i}{\partial t^2},$$

or (18)

$$F_i = \frac{1}{Q_i}\frac{\partial}{\partial q_i}\left(-\rho\frac{\partial^2 U}{\partial t^2}\right),$$

where ρ is the density. Thus the latter relationship, compared with that of $(18)_c$, gives

$$V = -\rho\frac{\partial^2 U}{\partial t^2},$$

however the general equation of vibratory motion, drawn from (19), is

$$\rho\frac{\partial^2 U}{\partial t^2} = A\Delta_2 U + 4\alpha B U. \tag{23}$$

Let us put, for considering a simple stationary vibration,

$$U = \Psi\cos\left(\frac{2\pi t}{\tau} + \mu\right) \tag{24}$$

where Ψ is a function only of the coordinates, and τ, μ are two constants, the first of which represents the period of a complete vibration, and the second the phase. Substituting this value of U in the equation (23) we get

$$A\Delta_2\Psi + 4\left(\frac{\pi^2\rho}{\tau^2} + \alpha B\right)\Psi = 0. \qquad (24)_a$$

When the curvature α is zero (*Euclidean* space), or positive (RIEMANNian space, or *spherical*), there is no admissible value of τ which nullifies the coefficient of Ψ. But when the curvature α is negative (GAUSSian space, or *pseudospherical*), that is when we have

$$\alpha = -\frac{1}{R^2},$$

where R is the radius of constant curvature, taking

$$\tau = \pi R\sqrt{\frac{\rho}{B}} \qquad (24)_b$$

the coefficient of Ψ becomes zero, and we obtain a singular class of vibrations, defined by

[1]

$$U = \Psi\cos\left(\frac{2t}{R}\sqrt{\frac{B}{\rho}} + \mu\right), \qquad (24)_c$$

by the which the function Ψ of the three coordinates q_i satisfies the equation

$$\Delta_2\Psi = 0 \qquad (24)_d$$

These vibrations, which are simultaneously free of rotation and dilatation, and which, as such, have no counterpart in ordinary space (except the so-called incompressible fluid), occur everywhere in the same direction as the force caused by the potential Ψ and have an amplitude proportional to this force. Such vibratory motion brings to birth internal tensions in the vibrating medium, the which one calculates with the formulae (5) and (11), as in the case of equilibrium, and all contain the periodic factor. Were we to take, for example, for Ψ, the values (21), (22) which satisfy the equation $(24)_d$, we would still find the tensions $(21)_a$, $(22)_a$, multiplied by the said factor.

If in equation (23) we suppose that U depend only upon ξ and upon t [where ξ has the same significance as in the equation (20)], we obtain the differential equation of spherical waves, under the form

$$\rho\frac{\partial^2 U}{\partial t^2} = \frac{A}{\sin^2(\xi\sqrt{\alpha})}\frac{\partial}{\partial\xi}\left\{\sin^2(\xi\sqrt{\alpha})\frac{\partial U}{\partial\zeta}\right\} + 4\alpha BU. \qquad (25)$$

[1] in Beltrami's Original: $(24)_d$

One satisfies this equation by putting

$$U = \frac{E \cos (g\xi + ht + k)}{\sin (\xi\sqrt{\alpha})} \tag{25$_a$}$$

where g, h, k, E are four constants, the first two of which are connected by the relation

$$h^2 = \frac{A}{\rho} g^2 - \frac{A + 4B}{\rho} \alpha. \tag{25$_b$}$$

We obtain thus progressive spherical waves, whose velocity of propagation

$$a = \pm \frac{h}{g}$$

and whose wavelength

$$\lambda = \pm \frac{2\pi}{g}$$

are connected by the relation

$$a^2 = \frac{A}{\rho} - \frac{A + 4B}{\rho} \frac{\alpha \lambda^2}{4\pi^2}. \tag{25$_c$}$$

Supposing $g^2 = \alpha$ one would return to the case considered above. These results, noted here with a haste for which I have to ask the reader to excuse me, seem to me such as to counsel that some attention be given to the new equations (17).

Pavia, 5 June, 1881.

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Appendix

q_i curvilinear othogonal coordinates of a point in 3d space

ds linear element

δq_i variation of the positions (displacements)

Q_i metrical coefficients (principal values of the metrical tensor)

$\nabla = Q_1\, Q_2\, Q_3$ (invariant of the metrical tensor)

$\theta_i = \frac{\partial \delta q_i}{\partial q_i} + \frac{\delta Q_i}{Q_i}$: dilatationes along coordinate axes q_i

ω_i shear deformation of face perpendicular to the axes

λ_i cosines of the angles which ds makes with the coordinate lines

n inward normal to the surface σ $(\cos(nq_i) = \vec{n} \cdot \vec{e_i} = \vec{n} \cdot \frac{\partial \vec{q}}{\partial q_i})$

dS element of volume

$d\sigma$ element of surface

F_i external force per volume

φ_i external force per surface

Θ_i, Ω_i unit tensions normal/tangential to the q_i axes

x_i total increments of the initial coordinates q_i by the deformation

Π deformation energy

A, B elastic material constants, as used by GREEN

$\vartheta = \theta_1 + \theta_2 + \theta_3$: cubic dilatation (relative volumetric expansion)

$\varpi = \omega_1^2 + \omega_2^2 + \omega_3^2 - 4(\theta_2\theta_3 + \theta_3\theta_1 + \theta_1\theta_2)$: change of shape

ϑ_i rotations (around the coordinate axes)

α (constant) curvature of space (with negative curvature: $\alpha = -\frac{1}{R^2}$)

r_{ij} radii of curvature ($\frac{1}{r_{ij}}$ ist the geodetic curvature of the line of intersection of the two surfaces $q_i =$ const., $q_j =$ const.)

α_i (non-constant) curvatures of surfaces $q_i =$ const.

ξ, η, ζ *spherical* coordinates of space with a constant curvature

u, z, ζ *cylindrical* coordinates of space with a constant curvature